

Categorical Results in the Theory of Two-Crossed Modules of Commutative Algebras

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Abstract

In this paper we explore some categorical results of 2-crossed module of commutative algebras extending work of Porter in [18]. We also show that the forgetful functor from the category of 2-crossed modules to the category of k -algebras, taking $\{L, M, P, \partial_2, \partial_1\}$ to the base algebra P , is fibred and cofibred considering the pullback (coinduced) and induced 2-crossed modules constructions, respectively. Also we consider free 2-crossed modules as an application of induced 2-crossed modules.

Introduction

Crossed modules of groups were initially defined by Whitehead [23, 24] as models for (homotopy) 2-types. The commutative algebra case of crossed modules is contained in the paper of Lichtenbaum and Schlessinger [16] and also the work of Gerstenhaber [13] under different names. Some categorical results and Koszul complex link are also given by Porter [18, 19]. Conduché, [12], in 1984 described the notion of 2-crossed modules as a model for 3-types. The commutative algebra version of 2-crossed modules has been defined by Grandjeán and Vale [14]. Arvasi and Porter [5],[6] have important studies related with that construction.

The purpose of this paper is to investigate some categorical theory of 2-crossed modules. It is considered the results as easy to prove but nonetheless giving some functorial relations between the category $\mathbf{X}_2\mathbf{Mod}$ of 2-crossed modules of commutative algebras and other categories with some adjoint pairs of functors would seem to be important to find out the question as to whether or not $\mathbf{X}_2\mathbf{Mod}$ is a fibred category.

Here we will give the construction of the pullback and induced 2-crossed modules of commutative algebras extending results of Shammu in [21] and Porter in [18] for crossed modules of algebras. The construction of pullback and induced 2-crossed modules will give us a pair of adjoint functors (ϕ^*, ϕ_*) where

$$\phi^* : \mathbf{X}_2\mathbf{Mod}/R \rightarrow \mathbf{X}_2\mathbf{Mod}/S \quad \text{and} \quad \phi_* : \mathbf{X}_2\mathbf{Mod}/S \rightarrow \mathbf{X}_2\mathbf{Mod}/R$$

with respect to an algebra morphism $\phi : S \rightarrow R$. Since we have pullback object of $\mathbf{X}_2\mathbf{Mod}$ along any arrow of $\mathbf{k}\text{-Alg}$, we get the fact that $\mathbf{X}_2\mathbf{Mod}$ is a fibred category and also cofibred which is the dual of the fibred.

We end with an application which leads to link free 2-crossed modules with induced 2-crossed modules.

Conventions

Throughout this paper k will be a fixed commutative ring and R will be a k -algebra with identity. All algebras will be commutative and actions will be left and the right actions in some references will be rewritten by using left actions.

1 Two-Crossed Modules of Algebras

Crossed modules of groups were initially defined by Whitehead [23, 24] as models for (homotopy) 2-types. Conduché, [12], in 1984 described the notion of 2-crossed module as a model for 3-types. Both crossed modules and 2-crossed modules have been adapted for use in the context of commutative algebras in [14, 19].

A crossed module is an algebra morphism $\partial : C \rightarrow R$ with an action of R on C satisfying $\partial(r \cdot c) = r\partial(c)$ and $\partial(c) \cdot c' = cc'$ for all $c, c' \in C, r \in R$. When the first equation is satisfied, ∂ is called pre-crossed module.

If (C, R, ∂) and (C', R', ∂') are crossed modules, a morphism,

$$(\theta, \varphi) : (C, R, \partial) \rightarrow (C', R', \partial'),$$

of crossed modules consists of k -algebra homomorphisms $\theta : C \rightarrow C'$ and $\varphi : R \rightarrow R'$ such that

$$(i) \partial'\theta = \varphi\partial \quad (ii) \theta(r \cdot c) = \varphi(r) \cdot \theta(c)$$

for all $r \in R, c \in C$. We thus get the category \mathbf{XMod} of crossed modules.

Examples of crossed modules are:

(i) Any ideal, I , in R gives an inclusion map $I \rightarrow R$, which is a crossed module then we will say (I, R, i) is an ideal pair. In this case, of course, R acts on I by multiplication and the inclusion homomorphism i makes (I, R, i) into a crossed module, an “inclusion crossed module”. Conversely, given any crossed module, $\partial : C \rightarrow R$ one easily sees that the image $\partial(C)$ of C is an ideal of R .

(ii) Any R -module M can be considered as an R -algebra with zero multiplication and hence the zero morphism $0 : M \rightarrow R$ sending everything in M to the zero element of R is a crossed module. Again conversely, any (C, R, ∂) crossed module, $\text{Ker}\partial$ is an ideal in C and inherits a natural R -module structure from R -action on C . Moreover, $\partial(C)$ acts trivially on $\text{Ker}\partial$, hence $\text{Ker}\partial$ has a natural $R/\partial(C)$ -module structure.

(iii) Let $M(R)$ be multiplication algebra defined by Mac Lane [15] (see also [17]) as the set of all multipliers $\delta : R \rightarrow R$ such that for all $r, r' \in R$, $\delta(rr') = r\delta(r')$ where R is a commutative k -algebra and $\text{Ann}(R) = 0$ or $R^2 = R$. Then $\mu : R \rightarrow M(R)$ is a crossed module given by $\mu(r) = \delta_r$ with $\delta_r(r') = rr'$ for all $r, r' \in R$. (See [2] for details).

(iv) Any epimorphism of algebras $C \rightarrow R$ with the kernel in the annihilator of C is a crossed module, with $r \in R$ acting on $c \in C$ by $r \cdot c = \bar{c}c$, where \bar{c} is any element in the pre-image of r .

Grandjeán and Vale [14] have given a definition of 2-crossed modules of algebras. The following is an equivalent formulation of that concept.

A 2-crossed module of k -algebras consists of a complex of P -algebras $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$ together with an action of P on all three algebras and a P -linear mapping

$$\{-, -\} : M \times M \rightarrow L$$

which is often called the Peiffer lifting such that the action of P on itself is by multiplication, ∂_2 and ∂_1 are P -equivariant.

$$\mathbf{PL1} : \partial_2 \{m_0, m_1\} = m_0 m_1 - \partial_1(m_1) \cdot m_0$$

$$\mathbf{PL2} : \{\partial_2(l_0), \partial_2(l_1)\} = l_0 l_1$$

$$\mathbf{PL3} : \{m_0, m_1 m_2\} = \{m_0 m_1, m_2\} + \partial_1(m_2) \cdot \{m_0, m_1\}$$

$$\mathbf{PL4} : \{m, \partial_2(l)\} + \{\partial_2(l), m\} = \partial_1(m) \cdot l$$

$$\mathbf{PL5} : \{m_0, m_1\} \cdot p = \{m_0 \cdot p, m_1\} = \{m_0, m_1 \cdot p\}$$

for all $m, m_0, m_1, m_2 \in M, l, l_0, l_1 \in L$ and $p \in P$.

Note that we have not specified that M acts on L . We could have done that as follows: if $m \in M$ and $l \in L$, define

$$m \cdot l = \{m, \partial_2(l)\}.$$

From this equation (L, M, ∂_2) becomes a crossed module. We can split **PL4** into two pieces:

PL4 :

$$(a) \quad \begin{aligned} \{m, \partial_2(l)\} &= m \cdot l \\ \{\partial_2(l), m\} &= m \cdot l - \partial_1(m) \cdot l. \end{aligned}$$

We denote such a 2-crossed module of algebras by $\{L, M, P, \partial_2, \partial_1\}$.

A morphism of 2-crossed modules is given by the following diagram

$$\begin{array}{ccccc} L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & P \\ f_2 \downarrow & & f_1 \downarrow & & f_0 \downarrow \\ L' & \xrightarrow{\partial'_2} & M' & \xrightarrow{\partial'_1} & P' \end{array}$$

where $f_0 \partial_1 = \partial'_1 f_1, f_1 \partial_2 = \partial'_2 f_2$

$$f_1(p \cdot m) = f_0(p) \cdot f_1(m) \quad , \quad f_2(p \cdot l) = f_0(p) \cdot f_2(l)$$

for all $m \in M, l \in L, p \in P$ and

$$\{-, -\}(f_1 \times f_1) = f_2 \{-, -\}$$

We thus get the category of 2-crossed modules denoting it by $\mathbf{X}_2\mathbf{Mod}$ and when the morphism f_0 above is the identity we will get $\mathbf{X}_2\mathbf{Mod}/P$ the category of 2-crossed modules over fixed algebra P .

Some remarks on Peiffer lifting of 2-crossed modules are: Suppose we have a 2-crossed module

$$L \xrightarrow{\partial_3} M \xrightarrow{\partial_1} P$$

with trivial Peiffer lifting. Then

(i) There is an action of P on L and M and the ∂ s are P -equivariant. (This gives nothing new in our special case.)

(ii) $\{-, -\}$ is a lifting of the Peiffer commutator so if $\{m, m'\} = 0$, the Peiffer identity holds for (M, P, ∂_1) , i.e. that is a crossed module.

(iii) if $l, l' \in L$, then $0 = \{\partial_2 l, \partial_2 l'\} = ll'$ and,

(iv) as $\{-, -\}$ is trivial $\partial_1(m) \cdot l = 0$ so ∂M has trivial action on L . Axioms PL3 and PL5 vanish.

The above remarks are known for 2-crossed modules of groups. These are handled in recent book of Porter in [20].

1.1 Functorial Relations with Some Other Categories

1. Let $M \xrightarrow{\partial} P$ be a pre-crossed module. The Peiffer ideal $\langle M, M \rangle$ is generated by the Peiffer commutators

$$\langle m, m' \rangle = \partial m \cdot m' - mm'$$

for all $m, m' \in M$. The pre-crossed modules in which all Peiffer commutators are trivial are precisely the crossed modules. Thus the category of crossed modules is the full subcategory of the category of pre-crossed modules whose objects are crossed modules. So we can define the following skeleton functor

$$Sk : \text{PXMod} \rightarrow \text{X}_2\text{Mod}$$

by $Sk(M, P, \partial_1) = \{\langle M, M \rangle, M, P, \partial_2, \partial_1\}$ as a 2-crossed module with the Peiffer lifting $\{m, m'\} = \langle m, m' \rangle$. This functor has a right adjoint truncation functor:

$$Tr : \text{X}_2\text{Mod} \rightarrow \text{PXMod}$$

given by $Tr\{L, M, P, \partial_2, \partial_1\} = (M, P, \partial_1)$.

2. Any crossed module gives a 2-crossed module. If (M, P, ∂) is a crossed module, the resulting sequence

$$L \rightarrow M \rightarrow P$$

is a 2-crossed module by taking $L = 0$. Thus we have

$$\alpha : \text{XMod} \rightarrow \text{X}_2\text{Mod}$$

defined by $\alpha(M, P, \partial) = \{0, M, P, 0, \partial\}$ that is the adjoint of the following functor:

$$\beta : \text{X}_2\text{Mod} \rightarrow \text{XMod}$$

given by $\beta\{L, M, P, \partial_2, \partial_1\} = (M/\text{Im } \partial_2, P, \partial_1)$ where $\text{Im } \partial_2$ is an ideal of M .

3. The functor $\delta : \text{XMod} \rightarrow \mathbf{k}\text{-Alg}$ which is given by $\delta(C, R, \partial) = R$ has a right adjoint $\gamma : \mathbf{k}\text{-Alg} \rightarrow \text{XMod}$, $\gamma(A) = (A, A, id_A)$.

Proposition 1 Given $X_2\text{Mod} \xrightleftharpoons[\alpha]{\beta} X\text{Mod} \xrightleftharpoons[\gamma]{\delta} k\text{-Alg}$ adjoint functors as defined above. Then $(\delta \circ \beta, \alpha \circ \gamma)$ is a pair of adjoint functors.

Proof. For $\{L, M, P, \partial_2, \partial_1\} \in X_2\text{Mod}$ and $R \in k\text{-Alg}$, we have functorial isomorphisms:

$$\begin{aligned} k\text{-Alg}(\delta\beta(\{L, M, P, \partial_2, \partial_1\}), R) &\simeq X\text{Mod}(\beta\{L, M, P, \partial_2, \partial_1\}, \gamma(R)) \\ &\simeq X_2\text{Mod}(\{L, M, P, \partial_2, \partial_1\}, \alpha\gamma(R)) \end{aligned}$$

■

Now we will give the construction of pullback and induced 2-crossed modules. Similar constructions have appeared in several studies on crossed module of groups, algebras and 2-crossed modules of groups, e.g. [1, 7, 8, 10, 11].

2 The Pullback Two-Crossed Modules

The construction of “change of base” is well-known in a module theory. The higher dimension of this had been considered by Porter [18] and Shammu [21]. There are called (co)-induced crossed modules. The first author and Gürmen were also deeply analysed that in [3]. In this section the functor that is going to be the right adjoint of the induced 2-crossed module, the “pullback” will be defined. This is an important construction which, given a morphism of algebras $\phi : S \rightarrow R$, enables us to change of base of 2-crossed modules.

Definition 2 Given a crossed module $\partial : C \rightarrow R$ and a morphism of k -algebras $\phi : S \rightarrow R$, the pullback crossed module can be given by

- (i) a crossed module $\phi^*(C, R, \partial) = (\partial^* : \phi^*(C) \rightarrow S)$
- (ii) given

$$(f, \phi) : (B, S, \mu) \longrightarrow (C, R, \partial)$$

crossed module morphism, then there is a unique (f^*, id_S) crossed module morphism that commutes the following diagram:

$$\begin{array}{ccc} & (B, S, \mu) & \\ & \swarrow (f^*, id_S) & \downarrow (f, \phi) \\ (\phi^*(C), S, \partial^*) & \xrightarrow{(\phi', \phi)} & (C, R, \partial) \end{array}$$

or more simply as

$$\begin{array}{ccc}
 B & \xrightarrow{f} & C \\
 \downarrow \mu & \swarrow f^* & \nearrow \phi' \\
 & \phi^*(C) & \\
 & \nwarrow \partial^* & \\
 S & \xrightarrow{\phi} & R \\
 & \downarrow \partial &
 \end{array}$$

where $\phi^*(C) = C \times_R S = \{(c, s) \mid \partial(c) = \phi(s)\}$, $\partial^*(c, s) = s$, $\phi'(c, s) = c$ for all $(c, s) \in \phi^*(C)$, and S acts on $\phi^*(C)$ via ϕ and the diagonal.

Definition 3 Given a 2-crossed module $\{C_2, C_1, R, \partial_2, \partial_1\}$ and a morphism of k -algebras $\phi : S \rightarrow R$, the pullback 2-crossed module can be given by

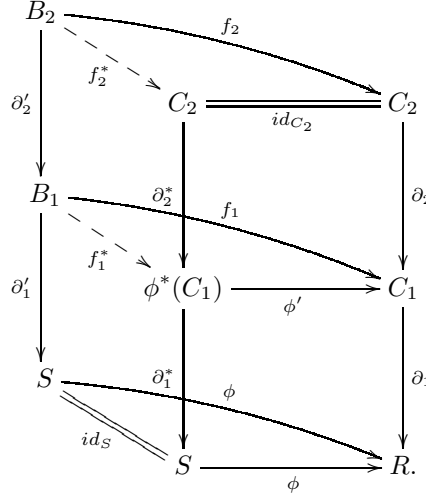
- (i) a 2-crossed module $\phi^*\{C_2, C_1, R, \partial_2, \partial_1\} = \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\}$
- (ii) given any morphism of 2-crossed modules

$$(f_2, f_1, \phi) : \{B_2, B_1, S, \partial_2', \partial_1'\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$$

there is a unique (f_2^*, f_1^*, id_S) 2-crossed module morphism that commutes the following diagram:

$$\begin{array}{ccc}
 & (B_2, B_1, S, \partial_2', \partial_1') & \\
 & \swarrow (f_2^*, f_1^*, id_S) & \downarrow (f_2, f_1, \phi) \\
 (C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*) & \xrightarrow{(id_{C_2}, \phi', \phi)} & (C_2, C_1, R, \partial_2, \partial_1)
 \end{array}$$

or more simply as



Let $\phi : S \rightarrow R$ be a morphism of k -algebras and let (C_1, R, ∂_1) be a pre-crossed module. We define

$$\phi^*(C_1) = C_1 \times_R S = \{(c_1, s) \mid \partial_1(c_1) = \phi(s)\}$$

which is usually pullback in the category of algebras. There is a commutative diagram

$$\begin{array}{ccc} \phi^*(C_1) & \xrightarrow{\phi'} & C_1 \\ \partial_1^* \downarrow & & \downarrow \partial_1 \\ S & \xrightarrow{\phi} & R \end{array}$$

where $\partial_1^*(c_1, s) = s$, $\phi'(c_1, s) = c_1$ and ∂_1^* is S -equivariant with the action $s' \cdot (c_1, s) = (\phi(s') \cdot c_1, s's)$ for all $(c_1, s) \in \phi^*(C_1)$ and $s \in S$.

So we get a pre-crossed module $(\phi^*(C_1), S, \partial_1^*)$ which is called the pullback pre-crossed module of (C_1, R, ∂_1) along ϕ . Then we can define a pullback of $\partial_2 : C_2 \rightarrow C_1$ along ϕ' as given in the following diagram

$$\begin{array}{ccc} \phi^*(C_2) & \xrightarrow{\phi''} & C_2 \\ \partial_2^* \downarrow & & \downarrow \partial_2 \\ \phi^*(C_1) & \xrightarrow{\phi'} & C_1 \end{array}$$

in which

$$\begin{aligned} \phi^*(C_2) &= \{(c_2, (c_1, s)) \mid \partial_2(c_2) = \phi'(c_1, s) = c_1, \phi(s) = \partial_1(c_1)\} \\ &= \{(c_2, (\partial_2(c_2), s)) \mid \phi(s) = \partial_1(\partial_2(c_2)) = 0\} \cong C_2 \times_{C_1} (Ker \partial_1 \times Ker \phi) \end{aligned}$$

for all $c_2 \in C_2$ and $(c_1, s) \in \phi^*(C_1)$.

Since pullback of a pullback is a pullback, we have already constructed the pullback composition

$$\phi^*(C_2) \xrightarrow{\partial_2^*} \phi^*(C_1) \xrightarrow{\partial_1^*} S$$

which is the pullback of $\partial_1 \partial_2 = 0$ by ϕ .

On the other hand, we can construct directly the pullback of $\partial_1 \partial_2 = 0$ by ϕ as $\partial : B \rightarrow S$ where $B = \{(c_2, s) \mid \phi(s) = 0\} \cong C_2 \times \text{Ker} \phi$ and $\partial(c_2, s) = s$. We can define the isomorphism $\Psi : \phi^*(C_2) \rightarrow B$, $\Psi(x) = (c_2, s)$ where $x = (c_2, (\partial_2(c_2), s)) \in \phi^*(C_2)$. So $\phi^*(C_2) \cong B$.

But, we find that the pullback $\phi^*(C_2) \xrightarrow{\partial_2^*} \phi^*(C_1) \xrightarrow{\partial_1^*} S$ is not a complex of S -algebras unless ϕ is a monomorphism. To see this, note that for $(c_2, s) \in C_2 \times \text{Ker} \phi$,

$$\partial_1^* \partial_2^*(c_2, s) = \partial_1^*(\partial_2 c_2, s) = s.$$

This last expression is equal to 0 if ϕ is a monomorphism. So $\phi^*(C_2) \cong C_2$.

Thus, we can give the pullback 2-crossed module of $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} R$ along ϕ as follows.

Proposition 4 *If $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} R$ is a 2-crossed module and if $\phi : S \rightarrow R$ is a monomorphism of k -algebras then*

$$C_2 \xrightarrow{\partial_2^*} \phi^*(C_1) \xrightarrow{\partial_1^*} S$$

is a pullback 2-crossed module where $\partial_2^(c_2) = (\partial_2(c_2), 0)$ and $\partial_1^*(c_1, s) = s$ and the action of S on $\phi^*(C_1)$ and C_2 by $s \cdot (c_1, s') = (\phi(s) \cdot c_1, ss')$ and $s \cdot c_2 = \phi(s) \cdot c_2$ respectively.*

Proof. Since

$$\begin{aligned} \partial_2^*(s \cdot c_2) &= (\partial_2(s \cdot c_2), 0) \\ &= (\partial_2(\phi(s) \cdot c_2), 0) \\ &= (\phi(s) \cdot \partial_2(c_2), 0) \\ &= s \cdot (\partial_2(c_2), 0) \\ &= s \cdot \partial_2^*(c_2), \end{aligned}$$

∂_2^* is S -equivariant. Also, we have seen above that ∂_1^* is S -equivariant and $C_2 \rightarrow \phi^*(C_1) \rightarrow S$ is a complex of S -algebras.

The Peiffer lifting

$$\{-, -\} : \phi^*(C_1) \times \phi^*(C_1) \rightarrow C_2$$

is given by $\{(c_1, s_1)(c'_1, s'_1)\} = \{c_1, c'_1\}$.

PL1:

$$\begin{aligned} (c_1, s_1)(c'_1, s'_1) - (c_1, s_1) \cdot \partial_1^*(c'_1, s'_1) &= (c_1 c'_1, s_1 s'_1) - (c_1, s_1) \cdot s'_1 \\ &= (c_1 c'_1, s_1 s'_1) - (c_1 \cdot \phi(s'_1), s_1 s'_1) \\ &= (c_1 c'_1 - c_1 \cdot \phi(s'_1), 0) \\ &= (c_1 c'_1 - c_1 \cdot \partial_1(c'_1), 0) \\ &= (\partial_2\{c_1, c'_1\}, 0) \\ &= \partial_2^*(\{c_1, c'_1\}) \\ &= \partial_2^*\{(c_1, s_1), (c'_1, s'_1)\}. \end{aligned}$$

PL2:

$$\begin{aligned}\{\partial_2^*(c_2), \partial_2^*(c'_2)\} &= \{(\partial_2(c_2), 0), (\partial_2(c'_2), 0)\} \\ &= \{\partial_2(c_2), \partial_2(c'_2)\} \\ &= c_2 c'_2.\end{aligned}$$

The rest of axioms of 2-crossed module is given in appendix.

$$(id_{C_2}, \phi', \phi) : \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$$

or diagrammatically,

$$\begin{array}{ccc} C_2 & \xrightarrow{id_{C_2}} & C_2 \\ \partial_2^* \downarrow & & \downarrow \partial_2 \\ \phi^*(C_1) & \xrightarrow{\phi'} & C_1 \\ \partial_1^* \downarrow & & \downarrow \partial_1 \\ S & \xrightarrow{\phi} & R \end{array}$$

is a morphism of 2-crossed modules. (See appendix.)

Suppose that

$$(f_2, f_1, \phi) : \{B_2, B_1, S, \partial'_2, \partial'_1\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$$

is any 2-crossed module morphism

$$\begin{array}{ccccc} B_2 & \xrightarrow{\partial'_2} & B_1 & \xrightarrow{\partial'_1} & S \\ f_2 \downarrow & & f_1 \downarrow & & \downarrow \phi \\ C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & R. \end{array}$$

Then we will show that there is a unique 2-crossed module morphism

$$(f_2^*, f_1^*, id_S) : \{B_2, B_1, S, \partial'_2, \partial'_1\} \rightarrow \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\}$$

$$\begin{array}{ccccc} B_2 & \xrightarrow{\partial'_2} & B_1 & \xrightarrow{\partial'_1} & S \\ f_2^* \downarrow & & f_1^* \downarrow & & \parallel id_S \\ C_2 & \xrightarrow{\partial_2} & \phi^*(C_1) & \xrightarrow{\partial_1} & S \end{array}$$

where $f_2^*(b_2) = f_2(b_2)$ and $f_1^*(b_1) = (f_1(b_1), \partial'_1(b_1))$ which is an element in $\phi^*(C_1)$. First, let us check that (f_2^*, f_1^*, id_S) is a morphism of 2-crossed modules.

For $b_1 \in B_1, b_2 \in B_2, s \in S$

$$\begin{aligned}f_2^*(s \cdot b_2) &= f_2(s \cdot b_2) \\ &= \phi(s) \cdot f_2(b_2) \\ &= \phi(s) \cdot f_2^*(b_2) \\ &= s \cdot f_2^*(b_2) \\ &= id_S(s) \cdot f_2^*(b_2),\end{aligned}$$

similarly $f_1^*(s \cdot b_1) = id_S(s) \cdot f_1^*(b_1)$, also the above diagram is commutative and

$$\begin{aligned}
 \{-, -\}(f_1^* \times f_1^*)(b_1, b'_1) &= \{-, -\}(f_1^*(b_1), f_1^*(b'_1)) \\
 &= \{-, -\}((f_1(b_1), \partial'_1(b_1)), (f_1(b'_1), \partial'_1(b'_1))) \\
 &= \{(f_1(b_1), \partial'_1(b_1)), (f_1(b'_1), \partial'_1(b'_1))\} \\
 &= \{f_1(b_1), f_1(b'_1)\} \\
 &= \{-, -\}(f_1(b_1), f_1(b'_1)) \\
 &= \{-, -\}(f_1 \times f_1)(b_1, b'_1) \\
 &= f_2\{-, -\}(b_1, b'_1) \\
 &= f_2(\{b_1, b'_1\}) \\
 &= f_2^*(\{b_1, b'_1\}) \\
 &= f_2^*\{-, -\}(b_1, b'_1)
 \end{aligned}$$

for all $b_1, b'_1 \in B_1$. So (f_2^*, f_1^*, id_S) is a morphism of 2-crossed modules.

Furthermore; following equations are easily verified:

$$id_{C_2} f_2^* = f_2 \quad \text{and} \quad \phi' f_1^* = f_1.$$

■

Thus we get a functor

$$\phi^* : \mathbf{X}_2\mathbf{Mod}/R \rightarrow \mathbf{X}_2\mathbf{Mod}/S$$

which gives our pullback 2-crossed module.

Remark 5 *These functors have the property that for any monomorphisms ϕ and ϕ' there are natural equivalences $\phi^* \phi'^* \simeq (\phi' \phi)^*$.*

2.1 The Examples of Pullback Two-Crossed Modules

Given 2-crossed module $\{\{0\}, I, R, 0, i\}$ where i is an inclusion of an ideal. The pullback 2-crossed module is

$$\begin{aligned}
 \phi^* \{\{0\}, I, R, 0, i\} &= \{\{0\}, \phi^*(I), S, \partial_2^*, \partial_1^*\} \\
 &= \{\{0\}, \phi^{-1}(I), S, \partial_2^*, \partial_1^*\}
 \end{aligned}$$

as,

$$\begin{aligned}
 \phi^*(I) &= \{(a, s) \mid \phi(s) = i(a) = a, s \in S, a \in I\} \\
 &\cong \{s \in S \mid \phi(s) = a\} = \phi^{-1}(I) \trianglelefteq S.
 \end{aligned}$$

The pullback diagram is

$$\begin{array}{ccc}
 \{0\} & \xlongequal{\quad} & \{0\} \\
 \partial_2^* = 0 \downarrow & & \downarrow 0 \\
 \phi^{-1}(I) & \xrightarrow{\quad} & I \\
 \partial_1^* \downarrow & & \downarrow i \\
 S & \xrightarrow[\phi]{} & R.
 \end{array}$$

Particularly if $I = \{0\}$, since ϕ is monomorphism we get

$$\phi^* (\{0\}) \cong \{s \in S \mid \phi(s) = 0\} = \ker \phi \cong \{0\}$$

and so $\{\{0\}, \{0\}, S, 0, 0\}$ is a pullback 2-crossed module.

Also if ϕ is an isomorphism and $I = R$, then $\phi^* (R) = R \times S$.

Similarly when we consider examples given in Section 1, the following diagrams are pullbacks.

$$\begin{array}{ccc} \{0\} & \xlongequal{\quad} & \{0\} \\ \downarrow & & \downarrow \\ \phi^*(R) & \longrightarrow & R \\ \downarrow & & \downarrow \\ M(S) & \xrightarrow{M(\phi)} & M(R) \end{array} \quad \begin{array}{ccc} \{0\} & \xlongequal{\quad} & \{0\} \\ \downarrow & & \downarrow \\ M \times \text{Ker} \phi & \longrightarrow & M \\ \downarrow & & \downarrow \\ S & \xrightarrow{\phi} & R \end{array}$$

3 The Induced Two-Crossed Modules

We will define a functor ϕ_* left adjoint to the pullback ϕ^* given in the previous section. The “induced 2-crossed module” functor ϕ_* is defined by the following universal property, developing works in [2] and [18].

Definition 6 For any crossed module $\mathcal{D} = (D, S, \partial)$ and any homomorphism $\phi : S \rightarrow R$ the crossed module induced by ϕ from ∂ should be given by:

- (i) a crossed module $\phi_*(\mathcal{D}) = (\phi_*(D), R, \phi_*\partial)$,
- (ii) a morphism of crossed modules $(f, \phi) : \mathcal{D} \rightarrow \phi_*(\mathcal{D})$, satisfying the dual universal property that for any morphism of crossed modules

$$(h, \phi) : \mathcal{D} \rightarrow \mathcal{B}$$

there is a unique morphism of crossed modules $h' : \phi_*(D) \rightarrow B$ such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{f} & \phi_*(D) \\ \downarrow \partial & & \downarrow \phi_*\partial \\ S & \xrightarrow{\phi} & R \end{array} \quad \begin{array}{c} \nearrow h \\ \nearrow h' \\ \searrow v \end{array}$$

commutes.

Definition 7 For any 2-crossed module $D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} S$ and a morphism $\phi : S \rightarrow R$ of k -algebras, the induced 2-crossed can be given by

- (i) a 2-crossed module $\phi_* \{D_2, D_1, S, \partial_2, \partial_1\} = \{\phi_*(D_2), \phi_*(D_1), R, \partial_{2*}, \partial_{1*}\}$
- (ii) given any morphism of 2-crossed modules

$$(f_2, f_1, \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}$$

then there is a unique (f_{2*}, f_{1*}, id_R) 2-crossed module morphism that commutes the following diagram:

$$\begin{array}{ccc}
 (D_2, D_1, S, \partial_2, \partial_1) & & \\
 \downarrow (f_2, f_1, \phi) & \searrow (\phi'', \phi', \phi) & \\
 (B_2, B_1, R, \partial'_2, \partial'_1) & \xleftarrow{(f_{2*}, f_{1*}, id_R)} & (\phi_*(D_2), \phi_*(D_1), R, \partial_{2*}, \partial_{1*})
 \end{array}$$

or more simply as

$$\begin{array}{ccccc}
 & & & & B_2 \\
 & & & \nearrow f_2 & \\
 D_2 & \xrightarrow{\phi''} & \phi_*(D_2) & \xrightarrow{f_{2*}} & B_2 \\
 \downarrow \partial_2 & & \downarrow \partial_{2*} & & \downarrow \partial'_2 \\
 D_1 & \xrightarrow{\phi'} & \phi_*(D_1) & \xrightarrow{f_{1*}} & B_1 \\
 \downarrow \partial_1 & & \downarrow \partial_{1*} & & \downarrow \partial'_1 \\
 S & \xrightarrow{\phi} & R & \xrightarrow{id_R} & R
 \end{array}$$

Proposition 8 Let $D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} S$ be a 2-crossed module and $\phi : S \rightarrow R$ be a morphism of k -algebras. Then $\phi_*(D_2) \xrightarrow{\partial_{2*}} \phi_*(D_1) \xrightarrow{\partial_{1*}} R$ is the induced 2-crossed module where $\phi_*(D_1)$ is generated as an algebra, by the set $D_1 \times R$ with defining relations

$$\begin{aligned}
 (d_1, r_1)(d'_1, r'_1) &= (d_1 d'_1, r_1 r'_1), \\
 (d_1, r) - (d'_1, r) &= (d_1 - d'_1, r), \\
 (s, d_1, r) &= (d_1, \phi(s) r)
 \end{aligned}$$

and $\phi_*(D_2)$ is generated as an algebra, by the set $D_2 \times R$ with defining relations

$$\begin{aligned} \{(d_1, r_1) + (d'_1, r'_1), (d''_1, r''_1)\} &= \{(d_1, r_1), (d''_1, r''_1)\} + \{(d'_1, r'_1), (d''_1, r''_1)\}, \\ \{(d_1, r_1), (d'_1, r'_1) + (d''_1, r''_1)\} &= \{(d_1, r_1), (d'_1, r'_1)\} + \{(d_1, r_1), (d''_1, r''_1)\}, \\ (d_2, r_2)(d'_2, r'_2) &= (d_2 d'_2, r_2 r'_2), \\ (d_2, r) + (d'_2, r) &= (d_2 + d'_2, r), \\ (s \cdot d_2, r) &= (d_2, \phi(s)r) \end{aligned}$$

for any $d_1, d'_1, d''_1 \in D_1$, $d_2, d'_2 \in D_2$, $s \in S$ and $r, r_1, r'_1, r''_1, r_2, r'_2 \in R$. The morphism $\partial_{2*} : \phi_*(D_2) \rightarrow \phi_*(D_1)$ is given by $\partial_{2*}(d_2, r) = (\partial_2(d_2), r)$ the action of $\phi_*(D_1)$ on $\phi_*(D_2)$ by $(d_1, r_1) \cdot (d_2, r_2) = (d_1 \cdot d_2, r_2)$, and the morphism $\partial_{1*} : \phi_*(D_1) \rightarrow R$ is given by $\partial_{1*}(d_1, r) = \phi \partial_1(d_1)r$, the action of R on $\phi_*(D_1)$ and $\phi_*(D_2)$ by $r \cdot (d_1, r_1) = (d_1, rr_1)$ and $r \cdot (d_2, r') = (d_2, rr')$ respectively.

Proof. (i) As $\partial_{1*}(\partial_{2*}(d_2, r)) = \partial_{1*}(\partial_2(d_2), r) = \phi(\partial_1(\partial_2(d_2)))r = \phi(0)r = 0$,

$$\phi_*(D_2) \xrightarrow{\partial_{2*}} \phi_*(D_1) \xrightarrow{\partial_{1*}} R$$

is a complex of k -algebras. The Peiffer lifting

$$\{-, -\} : \phi_*(D_1) \times \phi_*(D_1) \rightarrow \phi_*(D_2)$$

is given by $\{(d_1, r_1), (d'_1, r'_1)\} = (\{d_1, d'_1\}, r_1 r'_1)$ for all $(d_1, r_1), (d'_1, r'_1) \in \phi_*(D_1)$.

PL1:

$$\begin{aligned} \partial_{2*} \{(d_1, r_1), (d'_1, r'_1)\} &= \partial_{2*}(\{d_1, d'_1\}, r_1 r'_1) \\ &= (\partial_2 \{d_1, d'_1\}, r_1 r'_1) \\ &= (d_1 d'_1 - d_1 \cdot \partial_1(d'_1), r_1 r'_1) \\ &= (d_1 d'_1, r_1 r'_1) - (d_1 \cdot \partial_1(d'_1), r_1 r'_1) \\ &= (d_1 d'_1, r_1 r'_1) - (d_1, \phi(\partial_1(d'_1))) r_1 r'_1 \\ &= (d_1 d'_1, r_1 r'_1) - (d_1, r_1) \cdot (\phi(\partial_1(d'_1))) r'_1 \\ &= (d_1 d'_1, r_1 r'_1) - (d_1, r_1) \cdot \partial_{1*}(d'_1, r'_1) \\ &= (d_1, r_1)(d'_1, r'_1) - (d_1, r_1) \cdot \partial_{1*}(d'_1, r'_1). \end{aligned}$$

PL2:

$$\begin{aligned} \{\partial_{2*}(d_2, r_2), \partial_{2*}(d'_2, r'_2)\} &= \{(\partial_2(d_2), r_2), (\partial_2(d'_2), r'_2)\} \\ &= (\{\partial_2(d_2), \partial_2(d'_2)\}, r_2 r'_2) \\ &= (d_2 d'_2, r_2 r'_2) \\ &= (d_2, r_2)(d'_2, r'_2) \end{aligned}$$

for all $(d_1, r_1), (d'_1, r'_1) \in \phi_*(D_1)$, $(d_2, r_2), (d'_2, r'_2) \in \phi_*(D_2)$.

The rest of axioms of 2-crossed module is given in appendix.

(ii) It is clear that

$$(\phi'', \phi', \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{\phi_*(D_2), \phi_*(D_1), R, \partial_{2*}, \partial_{1*}\}$$

or diagrammatically,

$$\begin{array}{ccc}
 D_2 & \xrightarrow{\phi''} & \phi_*(D_2) \\
 \partial_2 \downarrow & & \downarrow \partial_{2*} \\
 D_1 & \xrightarrow{\phi'} & \phi_*(D_1) \\
 \partial_1 \downarrow & & \downarrow \partial_{1*} \\
 S & \xrightarrow{\phi} & R
 \end{array}$$

is a morphism of 2-crossed modules.

Suppose that

$$(f_2, f_1, \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}$$

is any 2-crossed module morphism. Then we will show that there is a 2-crossed module morphism

$$(f_{2*}, f_{1*}, id_R) : \{\phi_*(D_2), \phi_*(D_1), R, \partial_{2*}, \partial_{1*}\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}$$

$$\begin{array}{ccccc}
 \phi_*(D_2) & \xrightarrow{\partial_{2*}} & \phi_*(D_1) & \xrightarrow{\partial_{1*}} & R \\
 f_{2*} \downarrow & & f_{1*} \downarrow & & \parallel id_R \\
 B_2 & \xrightarrow{\partial'_2} & B_1 & \xrightarrow{\partial'_1} & R
 \end{array}$$

where $f_{2*}(d_2, r_2) = r_2 \cdot f_2(d_2)$ and $f_{1*}(d_1, r_1) = r_1 \cdot f_1(d_1)$. First we will check that (f_{2*}, f_{1*}, id_R) is a 2-crossed module morphism. We can see this easily as follows:

$$\begin{aligned}
 f_{2*}(r'_2 \cdot (d_2, r_2)) &= f_{2*}(d_2, r'_2 r_2) \\
 &= r'_2 \cdot (r_2 \cdot f_2(d_2)) \\
 &= r'_2 \cdot f_{2*}(d_2, r_2).
 \end{aligned}$$

Similarly $f_{1*}(r'_1 \cdot (d_1, r_1)) = r'_1 \cdot f_{1*}(d_1, r_1)$,

$$\begin{aligned}
 (f_{1*} \partial_{2*})(d_2, r_2) &= f_{1*}(\partial_{2*}(d_2, r_2)) \\
 &= f_{1*}(\partial_2(d_2), r_2) \\
 &= r_2 \cdot (f_1(\partial_2(d_2))) \\
 &= r_2 \cdot (\partial'_2(f_2(d_2))) \\
 &= \partial'_2(r_2 \cdot f_2(d_2)) \\
 &= (\partial'_2 f_{2*})(d_2, r_2)
 \end{aligned}$$

and $\partial'_1 f_{1*} = id_R \partial_{1*}$ for all $(d_1, r_1) \in \phi_*(D_1)$, $(d_2, r_2) \in \phi_*(D_2)$, $r'_1, r'_2 \in R$ and

$$\{-, -\}(f_{1*} \times f_{2*}) = f_2^* \{-, -\}.$$

■

So we get the induced 2-crossed module functor

$$\phi_* : \mathbf{X}_2\mathbf{Mod}/S \rightarrow \mathbf{X}_2\mathbf{Mod}/R.$$

We can give the following naturality condition for ϕ_* similar to remark 5.

Remark 9 They satisfy the “naturality condition” that there is a natural equivalence of functors

$$\phi'_* \phi_* \simeq (\phi' \phi)_*.$$

Theorem 10 For any morphism of k -algebras $\phi : S \rightarrow R$, ϕ_* is the left adjoint of ϕ^* .

Proof. From the naturality conditions given earlier, it is immediate that for any 2-crossed modules $\mathcal{D} = \{D_2, D_1, S, \partial_2, \partial_1\}$ and $\mathcal{B} = \{B_2, B_1, R, \partial_2, \partial_1\}$ there are bijections

$$(\mathbf{X}_2\text{Mod}/S)(\mathcal{D}, \phi^*(\mathcal{B})) \cong \{(f_2, f_1) \mid (f_2, f_1, \phi) : \mathcal{D} \rightarrow \mathcal{B} \text{ is a morphism in } \mathbf{X}_2\text{Mod}\}$$

as proved in proposition 4, and

$$(\mathbf{X}_2\text{Mod}/R)(\phi_*(\mathcal{D}), \mathcal{B}) \cong \{(f_2, f_1) \mid (f_2, f_1, \phi) : \mathcal{D} \rightarrow \mathcal{B} \text{ is a morphism in } \mathbf{X}_2\text{Mod}\}$$

as given in the proposition 8.

Their composition gives the bijection needed for adjointness. ■

Next if $\phi : S \rightarrow R$, is an epimorphism the induced 2-crossed module has a simpler description.

Proposition 11 Let $D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} S$ is a 2-crossed module, $\phi : S \rightarrow R$ is an epimorphism with $\text{Ker}\phi = K$. Then

$$\phi_*(D_2) \cong D_2/KD_2 \quad \text{and} \quad \phi_*(D_1) \cong D_1/KD_1,$$

where KD_2 denotes the ideal of D_2 generated by $\{k \cdot d_2 \mid k \in K, d_2 \in D_2\}$ and KD_1 denotes the ideal of D_1 generated by $\{k \cdot d_1 \mid k \in K, d_1 \in D_1\}$.

Proof. As $\phi : S \rightarrow R$ is an epimorphism, $R \cong S/K$. Since K acts trivially on $D_2/KD_2, D_1/KD_1$, $R \cong S/K$ acts on D_2/KD_2 by $r \cdot (d_2 + KD_2) = (s + K) \cdot (d_2 + KD_2) = s \cdot d_2 + KD_2$ and $R \cong S/K$ acts on D_1/KD_1 by $r \cdot (d_1 + KD_1) = (s + K) \cdot (d_1 + KD_1) = s \cdot d_1 + KD_1$.

$$D_2/KD_2 \xrightarrow{\partial_2^*} D_1/KD_1 \xrightarrow{\partial_1^*} R$$

is a 2-crossed module where

$$\partial_{2*}(d_2 + KD_2) = \partial_2(d_2) + KD_1, \partial_{1*}(d_1 + KD_1) = \partial_1(d_1) + K$$

and D_1/KD_1 acts on D_2/KD_2 by $(d_1 + KD_1) \cdot (d_2 + KD_2) = d_1 \cdot d_2 + KD_2$.
As

$$\partial_{1*}(\partial_{2*}(d_2 + KD_2)) = \partial_{1*}(\partial_2(d_2) + KD_1) = \partial_1(\partial_2(d_2)) + K = 0 + K \cong 0_R,$$

$D_2/KD_2 \xrightarrow{\partial_2^*} D_1/KD_1 \xrightarrow{\partial_1^*} R$ is a complex of k -algebras.

The Peiffer lifting

$$\{-, -\} : D_1/KD_1 \times D_1/KD_1 \rightarrow D_2/KD_2$$

is given by $\{d_1 + KD_1, d'_1 + KD_1\} = \{d_1, d'_1\} + KD_2$.

PL1:

$$\begin{aligned} & \partial_{2*} \{d_1 + KD_1, d'_1 + KD_1\} \\ = & \partial_{2*} (\{d_1, d'_1\} + KD_2) \\ = & \partial_2(\{d_1, d'_1\}) + KD_1 \\ = & (d_1 d'_1 - d_1 \cdot \partial_1(d'_1)) + KD_1 \\ = & (d_1 d'_1 + KD_1) - (d_1 \cdot \partial_1(d'_1) + KD_1) \\ = & (d_1 d'_1 + KD_1) - (d_1 + KD_1) \cdot (\partial_1(d'_1) + K) \\ = & (d_1 + KD_1)(d'_1 + KD_1) - (d_1 + KD_1) \cdot \partial_{1*}(d'_1 + KD_1). \end{aligned}$$

PL2:

$$\begin{aligned} \{\partial_{2*}(d_2 + KD_2), \partial_{2*}(d'_2 + KD_2)\} &= \{\partial_2(d_2) + KD_1, \partial_2(d'_2) + KD_1\} \\ &= \{\partial_2(d_2), \partial_2(d'_2)\} + KD_2 \\ &= d_2 d'_2 + KD_2 \\ &= (d_2 + KD_2)(d'_2 + KD_2). \end{aligned}$$

The rest of axioms of 2-crossed module is given by in appendix.

$$(\phi'', \phi', \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{D_2/KD_2, D_1/KD_1, R, \partial_{2*}, \partial_{1*}\}$$

or diagrammatically,

$$\begin{array}{ccc} D_2 & \xrightarrow{\phi''} & D_2/KD_2 \\ \partial_2 \downarrow & & \downarrow \partial_{2*} \\ D_1 & \xrightarrow{\phi'} & D_1/KD_1 \\ \partial_1 \downarrow & & \downarrow \partial_{1*} \\ S & \xrightarrow{\phi} & R \end{array}$$

is a morphism of 2-crossed modules. (See appendix.)

Suppose that

$$(f_2, f_1, \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}$$

is any 2-crossed module morphism. Then we will show that there is a unique 2-crossed module morphism

$$(f_{2*}, f_{1*}, id_R) : \{D_2/KD_2, D_1/KD_1, R, \partial_{2*}, \partial_{1*}\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}$$

$$\begin{array}{ccccc} D_2/KD_2 & \xrightarrow{\partial_{2*}} & D_1/KD_1 & \xrightarrow{\partial_{1*}} & R \\ f_{2*} \downarrow & & f_{1*} \downarrow & & \parallel id_R \\ B_2 & \xrightarrow{\partial'_2} & B_1 & \xrightarrow{\partial'_1} & R \end{array}$$

where $f_{2*}(d_2 + KD_2) = f_2(d_2)$ and $f_{1*}(d_1 + KD_1) = f_1(d_1)$. Since

$$f_2(k \cdot d_2) = \phi(k) \cdot f_2(d_2) = 0_R \cdot f_2(d_2) = 0_{B_2}$$

and similarly $f_1(d_1 + KD_1) = 0_{B_1}$, $f_2(KD_2) = 0_{B_2}$ and $f_1(KD_1) = 0_{B_1}$, f_{2*} and f_{1*} are well defined.

First let us check that (f_{2*}, f_{1*}, id_R) is a 2-crossed module morphism. For $d_2 + KD_2 \in D_2/KD_2$, $d_1 + KD_1 \in D_1/KD_1$ and $r \in R$,

$$\begin{aligned} f_{2*}(r \cdot (d_2 + KD_2)) &= f_{2*}((s + K) \cdot (d_2 + KD_2)) \\ &= f_{2*}((s \cdot d_2) + KD_2) \\ &= f_2(s \cdot d_2) \\ &= \phi(s) \cdot f_2(d_2) \\ &= s \cdot f_2(d_2) \\ &= (s + K) \cdot f_{2*}(d_2 + KD_2) \\ &= r \cdot f_{2*}(d_2 + KD_2) \end{aligned}$$

similarly $f_{1*}(r \cdot (d_1 + KD_1)) = r \cdot f_{1*}(d_1 + KD_1)$,

$$\begin{aligned} f_{1*}\partial_{2*}(d_2 + KD_2) &= f_{1*}(\partial_2(d_2) + KD_2) \\ &= f_1(\partial_2(d_2)) \\ &= \partial'_2(f_2(d_2)) \\ &= \partial'_2 f_{2*}(d_2 + KD_2) \end{aligned}$$

similarly $\partial'_1 f_{1*} = id_R \partial_{1*}$ and

$$\begin{aligned} f_{2*}\{-, -\}(d_1 + KD_1, d'_1 + KD_1) &= f_{2*}\{d_1 + KD_1, d'_1 + KD_1\} \\ &= f_{2*}(\{d_1, d'_1\} + KD_2) \\ &= f_2\{d_1, d'_1\} \\ &= f_2\{-, -\}(d_1, d'_1) \\ &= \{-, -\}(f_1 \times f_1)(d_1, d'_1) \\ &= \{f_1(d_1), f_1(d'_1)\} \\ &= \{f_{1*}(d_1 + KD_1), f_{1*}(d'_1 + KD_1)\} \\ &= \{-, -\}(f_{1*} \times f_{1*})(d_1 + KD_1, d'_1 + KD_1). \end{aligned}$$

So (f_{2*}, f_{1*}, id_R) is a morphism of 2-crossed modules. Furthermore; following equations are verified.

$$f_{2*}\phi'' = f_2 \text{ and } f_{1*}\phi' = f_1.$$

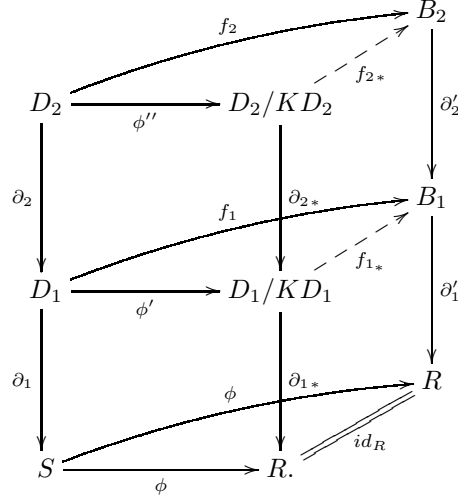
So given any morphism of 2-crossed modules

$$(f_2, f_1, \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}$$

then there is a unique (f_{2*}, f_{1*}, id_R) 2-crossed module morphism that commutes the following diagram:

$$\begin{array}{ccc} (D_2, D_1, S, \partial_2, \partial_1) & & \\ \downarrow (f_2, f_1, \phi) & \searrow (\phi'', \phi', \phi) & \\ (B_2, B_1, R, \partial'_2, \partial'_1) & \xleftarrow[(f_{2*}, f_{1*}, id_R)]{-- -- --} & (D_2/KD_2, D_1/KD_1, R, \partial_{2*}, \partial_{1*}) \end{array}$$

or more simply as



■

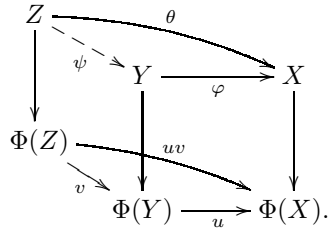
4 Fibrations and Cofibrations of Categories

The notion of fibration of categories is intended to give a general background to constructions analogous to pullback by a morphism. It seems to be a very useful notion for dealing with hierarchical structures. A functor which forgets the top level of structure is often usefully seen as a fibration or cofibration of categories.

We rewrite from [9] and [22] the definition of fibration and cofibration of categories and some propositions.

Definition 12 Let $\Phi : \mathbf{X} \rightarrow \mathbf{B}$ be a functor. A morphism $\varphi : Y \rightarrow X$ in \mathbf{X} over $u := \Phi(\varphi)$ is called *cartesian* if and only if for all $v : Z \rightarrow Y$ in \mathbf{B} and $\theta : Z \rightarrow X$ with $\Phi(\theta) = uv$ there is a unique morphism $\psi : Z \rightarrow Y$ with $\Phi(\psi) = v$ and $\theta = \varphi\psi$.

This is illustrated by the following diagram:



If $Y \rightarrow X$ is a cartesian arrow of \mathbf{X} mapping to an arrow $\Phi(Y) \rightarrow \Phi(X)$ of \mathbf{B} , we also say that Y is a *pullback* of X to $\Phi(Y)$

A morphism $\alpha : Z \rightarrow Y$ is called *vertical* (with respect to Φ) if and only if $\Phi(\alpha)$ is an identity morphism in \mathbf{B} . In particular, for $I \in \mathbf{B}$ we write \mathbf{X}/I , called the *fibre over I* , for the subcategory of \mathbf{X} consisting of those morphisms α with $\Phi(\alpha) = id_I$.

Definition 13 The functor $\Phi : \mathbf{X} \rightarrow \mathbf{B}$ is a *fibration* or *category fibred over \mathbf{B}* if and only if for all $u : J \rightarrow I$ in \mathbf{B} and $X \in \mathbf{X}/I$ there is a cartesian morphism $\varphi : Y \rightarrow X$ over u . Such a φ is called a *cartesian lifting* of X along u .

In other words, in a category fibred over \mathbf{B} , $\Phi : \mathbf{X} \rightarrow \mathbf{B}$, we can pull back objects of \mathbf{X} along any arrow of \mathbf{B} .

Notice that cartesian liftings of $X \in \mathbf{X}/I$ along $u : J \rightarrow I$ are unique up to vertical isomorphism: given two pullbacks $\varphi : Y \rightarrow X$ and $\bar{\varphi} : \bar{Y} \rightarrow X$ of X to I , the unique arrow $\theta : \bar{Y} \rightarrow Y$ that fits into the diagram

$$\begin{array}{ccccc} \bar{Y} & & & & \\ \downarrow & \searrow & & \searrow & \\ & Y & \longrightarrow & X & \\ & \downarrow & & \downarrow & \\ J & & & & I \\ \parallel & \nearrow & & \nearrow & \\ & J & \longrightarrow & I & \end{array}$$

is an isomorphism; the inverse is the arrow $Y \rightarrow \bar{Y}$ obtained by exchanging Y and \bar{Y} in the diagram above. In other words, a pullback is unique, up to a unique isomorphism.

The following results in the case of crossed modules of groupoids and commutative algebras have appeared in [9] and [3], respectively.

Proposition 14 The forgetful functor $p : \mathbf{X}_2\mathbf{Mod} \rightarrow \mathbf{k}\text{-Alg}$ which sends $\{C_2, C_1, R, \partial_2, \partial_1\} \mapsto R$ is fibred.

Proof. It is enough to consider the pullback construction from proposition 4 to prove that p is a fibred. Thus the morphism $(id_{C_2}, \phi', \phi) : \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$ of $\mathbf{X}_2\mathbf{Mod}$ is cartesian. Because for any morphism

$$(f_2, f_1, \phi) : \{B_2, B_1, S, \partial_2', \partial_1'\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$$

in $\mathbf{X}_2\mathbf{Mod}$ and any morphism

$$id_S : p(\{B_2, B_1, S, \partial_2', \partial_1'\}) \rightarrow p(\{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\})$$

in $\mathbf{k}\text{-Alg}$ with $p(id_{C_2}, \phi', \phi) \circ id_S = p(f_2, f_1, \phi)$, there exists a unique arrow (f_2^*, f_1^*, id_S) with $p(f_2^*, f_1^*, id_S) = id_S$ and $(id_{C_2}, \phi', \phi) \circ (f_2^*, f_1^*, id_S) = (f_2, f_1, \phi)$

as in the commutative diagram

$$\begin{array}{ccccc}
 \{B_2, B_1, S, \partial'_2, \partial'_1\} & \xrightarrow{(f_2, f_1, \phi)} & \{C_2, C_1, R, \partial_2, \partial_1\} \\
 \downarrow p & \nearrow (f_2^*, f_1^*, id_S) & \downarrow p & \xrightarrow{(id_{C_2}, \phi', \phi)} & \downarrow p \\
 S & \xrightarrow{\phi} & S & \xrightarrow{\phi} & R
 \end{array}$$

■

In considering the functor $p : \mathbf{X}_2\mathbf{Mod} \rightarrow \mathbf{k}\text{-Alg}$ as a fibration, if we fix the monomorphism $\phi : S \rightarrow R$ in $\mathbf{k}\text{-Alg}$, the cartesian lifting $\{C_2, \phi^*(C_1), R, \partial_2, \partial_1\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$ of $\{C_2, C_1, R, \partial_2, \partial_1\}$ along ϕ for $\{C_2, C_1, R, \partial_2, \partial_1\} \in \mathbf{X}_2\mathbf{Mod}/R$, gives a so-called reindexing functor

$$\phi^* : \mathbf{X}_2\mathbf{Mod}/R \rightarrow \mathbf{X}_2\mathbf{Mod}/S$$

given in section 2 and defined as objects by $\{C_2, C_1, R, \partial_2, \partial_1\} \mapsto \{C_2, \phi^*(C_1), R, \partial_2, \partial_1\}$ and the image of a morphism $\phi^*(\alpha, \beta, id_R) = (\alpha, \phi^*(\beta), id_S)$ the unique arrow commuting the quadrangles in the following diagram:

$$\begin{array}{ccccc}
 C_2 & \xrightarrow{\alpha} & C_2 & \xrightarrow{\alpha} & C_2' \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
 \phi^*(C_1) & \xrightarrow{\phi^*(\beta)} & C_1 & \xrightarrow{\beta} & C_1' \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\
 S & \xrightarrow{id_S} & S & \xrightarrow{id_R} & R
 \end{array}$$

We can use this reindexing functor to get an adjoint situation for each monomorphism $\phi : S \rightarrow R$ in $\mathbf{k}\text{-Alg}$.

Proposition 15 *Let $p : \mathbf{X}_2\mathbf{Mod} \rightarrow \mathbf{k}\text{-Alg}$ be the forgetful functor with $p\{C_2, C_1, R, \partial_2, \partial_1\} \mapsto R$, monomorphism $\phi : S \rightarrow R$ be in $\mathbf{k}\text{-Alg}$, and $\phi^* : \mathbf{X}_2\mathbf{Mod}/R \rightarrow \mathbf{X}_2\mathbf{Mod}/S$ be reindexing functor, pullback. Then there is a bijection*

$$\mathbf{X}_2\mathbf{Mod}/S(\mathcal{B}, \phi^*(\mathcal{C})) \cong \mathbf{X}_2\mathbf{Mod}/\phi(\mathcal{B}, \mathcal{C})$$

natural in $\mathcal{B} \in \mathbf{X}_2\mathbf{Mod}/S$, $\mathcal{C} \in \mathbf{X}_2\mathbf{Mod}/R$ where $\mathcal{B} = \{B_2, B_1, S, \partial'_2, \partial'_1\}$, $\mathcal{C} = \{C_2, C_1, R, \partial_2, \partial_1\}$ and $\mathbf{X}_2\mathbf{Mod}/\phi(\mathcal{B}, \mathcal{C})$ consists of those morphisms $(f_2, f_1, \phi) \in \mathbf{X}_2\mathbf{Mod}(\mathcal{B}, \mathcal{C})$ with $p(f_2, f_1, \phi) = \phi$.

Proof. Since $(id_{C_2}, \phi', \phi) : \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$ is a cartesian morphism over ϕ as may be seen with the proof of proposition 14, there is a unique morphism

$$\Psi : \{B_2, B_1, S, \partial_2', \partial_1'\} \rightarrow \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\}$$

with $p\Psi = id_S$ for $(f_2, f_1, \phi) : \{B_2, B_1, S, \partial_2', \partial_1'\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$ with $p((f_2, f_1, \phi)) = \phi$.

On the other hand; it is clear that there is a morphism

$$\phi' \circ \gamma : \{B_2, B_1, S, \partial_2', \partial_1'\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$$

for $\gamma : \{B_2, B_1, S, \partial_2', \partial_1'\} \rightarrow \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\}$. ■

For composable monomorphism $\phi : S \rightarrow R$ and $\phi' : R \rightarrow T$, there is a natural equivalence

$$c_{\phi', \phi} : \phi^* \phi'^* \cong (\phi' \phi)^*$$

but not equality as shown in the following diagram in which the morphisms $(id_{C_2}, h, \phi' \phi)$, (id_{C_2}, g, ϕ) and (id_{C_2}, g', ϕ') are cartesian and so the composition $(id_{C_2}, g, \phi) \circ (id_{C_2}, g', \phi')$ is cartesian and (id_{C_2}, k, id_S) is the unique vertical morphism from proposition 14 making the diagram commute:

$$\begin{array}{ccccccc}
 C_2 & & & & & & \\
 \downarrow & \swarrow id_{C_2} & & & \searrow id_{C_2} & & \\
 & C_2 & \xrightarrow{id_{C_2}} & C_2 & \xrightarrow{id_{C_2}} & C_2 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 (\phi' \phi)^*(C_1) & \xrightarrow{h} & C_2 & \xrightarrow{g} & \phi'^*(C_1) & \xrightarrow{g'} & C_1 \\
 & \swarrow k & & & \searrow & & \\
 & \phi^* \phi'^*(C_1) & \xrightarrow{g} & \phi'^*(C_1) & \xrightarrow{g'} & C_1 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 S & \xrightarrow{\phi' \phi} & R & \xrightarrow{\phi'} & T & & \\
 & \swarrow id_S & & & \searrow & & \\
 & S & \xrightarrow{\phi} & R & \xrightarrow{\phi'} & T &
 \end{array}$$

Definition 16 Let $\Phi : \mathbf{X} \rightarrow \mathbf{B}$ be a functor. A morphism $\psi : Z \rightarrow Y$ in \mathbf{X} over $v := \Phi(\psi)$ is called *cocartesian* if and only if for all $u : J \rightarrow I$ in \mathbf{B} and $\theta : Z \rightarrow X$ with $\Phi(\theta) = uv$ there is a unique morphism $\varphi : Y \rightarrow X$ with

$\Phi(\varphi) = u$ and $\theta = \varphi\psi$. This is illustrated by the following diagram:

$$\begin{array}{ccccc}
 & & \theta & \nearrow & X \\
 & & \psi & \xrightarrow{\quad} & Y \\
 Z & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow & & \downarrow \\
 \Phi(Z) & \xrightarrow{\quad} & \Phi(Y) & \xrightarrow{\quad} & \Phi(X)
 \end{array}$$

$\swarrow \theta$ $\searrow \varphi$ $\swarrow uv$ $\searrow u$

The functor $\Phi : \mathbf{X} \rightarrow \mathbf{B}$ is a cofibration or category cofibred over \mathbf{B} if and only if for all $v : K \rightarrow J$ in \mathbf{B} and $Z \in \mathbf{X}/K$ there is a cocartesian morphism $\psi : Z \rightarrow Z'$ over v . Such a ψ is called a cocartesian lifting of Z along v .

The cocartesian liftings of $Z \in \mathbf{X}/K$ along $v : K \rightarrow J$ are also unique up to vertical isomorphism.

It is interesting to get a characterisation of the cofibration property for a functor that already is a fibration. The following is a useful weakening of the condition for cocartesian in the case of a fibration of categories.

Proposition 17 *Let $\Phi : \mathbf{X} \rightarrow \mathbf{B}$ be a fibration of categories. Then $\psi : Z \rightarrow Y$ in \mathbf{X} over $v : K \rightarrow J$ in \mathbf{B} is cocartesian if and only if for all $\theta' : Z \rightarrow X'$ over v there is a unique morphism $\psi' : Y \rightarrow X'$ in \mathbf{X}/J with $\theta' = \psi'\psi$.*

If we take the fibration $p : \mathbf{X}_2\text{Mod} \rightarrow \mathbf{k}\text{-Alg}$ and reindexing functor $\phi^* : \mathbf{X}_2\text{Mod}/R \rightarrow \mathbf{X}_2\text{Mod}/S$ for monomorphism $\phi : S \rightarrow R$, we get the morphism

$$\phi_{\{D_2, D_1, S, \partial_2, \partial_1\}} : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \phi_* \{D_2, D_1, S, \partial_2, \partial_1\}$$

which is cocartesian over ϕ , for all $\{D_2, D_1, S, \partial_2, \partial_1\} \in \mathbf{X}_2\text{Mod}/S$.

Because there is the functor $\phi_* : \mathbf{X}_2\text{Mod}/S \rightarrow \mathbf{X}_2\text{Mod}/R$ which is left adjoint to ϕ^* as mentioned in theorem 10, and also by proposition 17, the adjointness gives the bijection required for cocartesian property. Thus $\phi_{\{D_2, D_1, S, \partial_2, \partial_1\}}$ is cocartesian over ϕ .

So, by constructing the adjoint ϕ_* of ϕ^* for ϕ , the fibration p is also a cofibration.

5 Application: Free 2-Crossed Modules

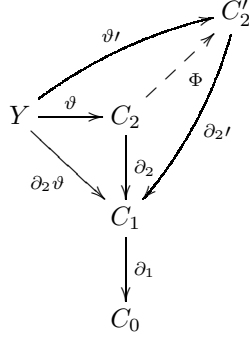
The definition of a free 2-crossed module is similar in some ways to the corresponding definition of a free crossed module. We recall the definition of a free crossed module and a free 2-crossed module from [4].

Let (C, R, ∂) be a pre-crossed module, let Y be a set and let $\vartheta : Y \rightarrow C$ be a function, then (C, R, ∂) is said to be a free pre-crossed module with basis ϑ or, alternatively, on the function $\partial\vartheta : Y \rightarrow R$ if for any pre-crossed module (C', R, ∂') and function $\vartheta' : Y \rightarrow C'$ such that $\partial'\vartheta' = \partial\vartheta$, there is a unique morphism

$$\phi : (C, R, \partial) \rightarrow (C', R, \partial')$$

such that $\phi\vartheta = \vartheta'$.

Let $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ be a 2-crossed module, let Y be a set and let $\vartheta : Y \rightarrow C_2$ be a function, then $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is said to be a free 2-crossed module with basis ϑ or, alternatively, on the function $\partial_2\vartheta : Y \rightarrow C_1$, if for any 2-crossed module $\{C'_2, C_1, C_0, \partial'_2, \partial_1\}$ and function $\vartheta' : Y \rightarrow C'_2$ such that $\partial_2\vartheta = \partial'_2\vartheta'$, there is a unique morphism $\Phi : C_2 \rightarrow C'_2$ such that $\partial'_2\Phi = \partial_2$.



The following proposition is an application of induced 2-crossed modules using the universal properties of free 2-crossed modules and of universal morphism of 2-crossed modules.

Proposition 18 *Suppose $\phi : S \rightarrow R$ is a k -algebra morphism, then the 2-crossed module $\{C_2, C_1, R, \partial_2, \partial_1\}$ is the free 2-crossed module on $\partial_2\vartheta$ with the morphism $\vartheta : Y \rightarrow C_2$ if and only if the morphism $(\vartheta, \partial_2\vartheta, \phi) : \{Y, Y, S, id_Y, 0\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$ of 2-crossed modules is a universal morphism, i.e. for any 2-crossed module morphism*

$$(\vartheta', \partial_2\vartheta, \phi) : \{Y, Y, S, id_Y, 0\} \rightarrow \{C'_2, C_1, R, \partial'_2, \partial_1\}$$

there exists a unique (Φ, id_{C_1}, id_R) 2-crossed module morphism such that $\Phi\vartheta = \vartheta'$.

Proof. Suppose $(\vartheta, \partial_2\vartheta, \phi)$ is a universal morphism of 2-crossed modules. Let

$$(\vartheta', \partial_2\vartheta, \phi) : \{Y, Y, S, id_Y, 0\} \rightarrow \{C'_2, C_1, R, \partial'_2, \partial_1\}$$

be a 2-crossed module morphism. Then there exists a unique

$$(\Phi, id_{C_1}, id_R) : \{C_2, C_1, R, \partial_2, \partial_1\} \rightarrow \{C'_2, C_1, R, \partial'_2, \partial_1\}$$

2-crossed module morphism such that $\Phi\vartheta = \vartheta'$. This description gives the required free 2-crossed module $\{C_2, C_1, R, \partial_2, \partial_1\}$ on $\partial_2\vartheta$.

On the other hand, let $\{C_2, C_1, R, \partial_2, \partial_1\}$ be a free 2-crossed module on $\partial_2\vartheta$, $\{C'_2, C_1, R, \partial'_2, \partial_1\}$ be a 2-crossed module and $\vartheta' : Y \rightarrow C'_2$ be an algebra morphism such that $\partial_2\vartheta = \partial'_2\vartheta'$. Then by using the universal property of a free

2-crossed module, we get a unique morphism $\Phi : C_2 \rightarrow C_2'$ such that $\partial_2' \Phi = \partial_2$. This proves that

$$(\vartheta, \partial_2 \vartheta, \phi) : \{Y, Y, S, id_Y, 0\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$$

is a universal morphism of 2-crossed modules.

■ This proposition leads to link free 2-crossed modules with induced 2-crossed modules.

Given any k -algebra morphism $\phi : S \rightarrow R$, we note that $Y \xrightarrow{id_Y} Y \xrightarrow{0} S$ is a 2-crossed module and form the induced 2-crossed module $\{\phi_*(Y), \phi_*(Y), R, id_{Y*}, 0_*\}$ as described in section 3.

Proposition 18 implies that the free 2-crossed module on the morphism $Y \rightarrow \phi_*(Y)$ is the induced 2-crossed module on ϕ .

Note that the definition of free 2-crossed modules has been chosen to tie in with the definition of freeness given in other more general context.

Proposition 19 *Let $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ be a free 2-crossed module on f and $f = 0$, then C_2 is a free C_1 -module on f .*

Proof. Given any free 2-crossed module $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ on f , then (C_2, C_1, ∂_2) is a free crossed module on f . Since (C_2, C_1, ∂_2) is free on f , then there exists a function $\vartheta : Y \rightarrow C_2$ such that $\partial_2 \vartheta = f = 0$. Let C_2' be a C_1 -module and $\vartheta' : Y \rightarrow C_2'$ be a function into C_2' . Form the crossed module $(C_2', C_1, 0)$. Since $\partial_2 \vartheta = f = 0 = 0\vartheta'$, thus there is a unique morphism $\Phi : C_2 \rightarrow C_2'$ such that $\Phi \vartheta = \vartheta'$. Therefore C_2 is free C_1 -module on f . ■

6 Appendix

The proof of proposition 4

PL3:

$$\begin{aligned}
 & \{(c_1, s), (c'_1, s') (c''_1, s'')\} \\
 = & \{(c_1, s), (c'_1 c''_1, s' s'')\} \\
 = & \{c_1, c'_1 c''_1\} \\
 = & \{c_1 c'_1, c''_1\} + \partial_1 (c''_1) \cdot \{c_1, c'_1\} \\
 = & \{c_1 c'_1, c''_1\} + \phi (s'') \cdot \{c_1, c'_1\} \\
 = & \{c_1 c'_1, c''_1\} + s'' \cdot \{c_1, c'_1\} \\
 = & \{c_1 c'_1, c''_1\} + \partial_1^* (c''_1, s'') \cdot \{c_1, c'_1\} \\
 = & \{(c_1 c'_1, s s'), (c''_1, s'')\} + \partial_1^* (c''_1, s'') \cdot \{(c_1, s), (c'_1, s')\} \\
 = & \{(c_1, s) (c'_1, s'), (c''_1, s'')\} + \partial_1^* (c''_1, s'') \cdot \{(c_1, s), (c'_1, s')\}.
 \end{aligned}$$

PL4:

$$\begin{aligned}
 & \{(c_1, s), \partial_2^* (c_2)\} + \{\partial_2^* (c_2), (c_1, s)\} \\
 = & \{(c_1, s), (\partial_2 (c_2), 0)\} + \{(\partial_2 (c_2), 0), (c_1, s)\} \\
 = & \{c_1, \partial_2 (c_2)\} + \{\partial_2 (c_2), c_1\} \\
 = & \partial_1 (c_1) \cdot c_2 \\
 = & \phi (s) \cdot c_2 \\
 = & s \cdot c_2 \\
 = & \partial_1^* (c_1, s) \cdot c_2.
 \end{aligned}$$

PL5:

$$\begin{aligned}
 \{(c_1, s) \cdot s'', (c'_1, s')\} &= \{(c_1 \cdot \phi(s''), s s''), (c'_1, s')\} \\
 &= \{c_1 \cdot \phi(s''), c'_1\} \\
 &= \{c_1, c'_1\} \cdot \phi(s'') \\
 &= \{c_1, c'_1\} \cdot s'' \\
 &= \{(c_1, s), (c'_1, s')\} \cdot s''. \\
 \{(c_1, s), (c'_1, s') \cdot s''\} &= \{(c_1, s) (c'_1 \cdot \phi(s''), s' s'')\} \\
 &= \{c_1, c'_1 \cdot \phi(s'')\} \\
 &= \{c_1, c'_1\} \cdot \phi(s'') \\
 &= \{c_1, c'_1\} \cdot s'' \\
 &= \{(c_1, s), (c'_1, s')\} \cdot s''.
 \end{aligned}$$

for all $(c_1, s), (c'_1, s'), (c''_1, s'') \in \phi^*(C_1), c_2 \in C_2$ and $s'' \in S$.

Let us check that $(id_{C_2}, \phi', \phi) : \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\} \rightarrow \{C_2, C_1, R, \partial_2, \partial_1\}$ is a morphism of 2-crossed modules where $\phi'(c_1, s) = c_1$.

$$\begin{aligned}
 id_{C_2} (s \cdot c_2) &= s \cdot c_2 \\
 &= \phi (s) \cdot c_2 \\
 &= \phi (s) \cdot id_{C_2} (c_2)
 \end{aligned}$$

and similarly $\phi'(s \cdot (c_1, s')) = \phi (s) \cdot \phi'(c_1, s')$

$$\begin{aligned}
 (\phi' \partial_2^*) (c_2) &= \phi' (\partial_2 (c_2), 0) \\
 &= \partial_2 (c_2) \\
 &= \partial_2 id_{C_2} (c_2)
 \end{aligned}$$

and $\phi\partial_1^* = \partial_1\phi'$ for all $(c_1, s') \in \phi^*(C_1)$, $c_2 \in C_2$ and $s \in S$.

$$\begin{aligned}
 id_{C_2} \{-, -\}((c_1, s), (c'_1, s')) &= id_{C_2}(\{(c_1, s), (c'_1, s')\}) \\
 &= \{c_1, c'_1\} \\
 &= \{-, -\}(c_1, c'_1) \\
 &= \{-, -\}(\phi'(c_1, s), \phi'(c'_1, s')) \\
 &= \{-, -\}(\phi' \times \phi')((c_1, s), (c'_1, s'))
 \end{aligned}$$

for all $(c_1, s), (c'_1, s') \in \phi^*(C_1)$.

The proof of proposition 8

PL3:

$$\begin{aligned}
 &\{(d_1, r), (d'_1, r')(d''_1, r'')\} \\
 &= \{(d_1, r), (d'_1 d''_1, r' r'')\} \\
 &= (\{d_1, d'_1 d''_1\}, r(r' r'')) \\
 &= (\{d_1 d'_1, d''_1\} + \partial_1(d''_1) \cdot \{d_1, d'_1\}, r(r' r'')) \\
 &= (\{d_1 d'_1, d''_1\}, r(r' r'')) + (\partial_1(d''_1) \cdot \{d_1, d'_1\}, r(r' r'')) \\
 &= (\{d_1 d'_1, d''_1\}, r r' r'') + (\{d_1, d'_1\}, \phi(\partial_1(d''_1)) r r' r'') \\
 &= (\{d_1 d'_1, d''_1\}, r r' r'') + (\phi(\partial_1(d''_1)) r'') \cdot (\{d_1, d'_1\}, r r') \\
 &= (\{d_1 d'_1, d''_1\}, r r' r'') + \partial_{1*}(d''_1, r'') \cdot (\{d_1, d'_1\}, r r') \\
 &= \{(d_1 d'_1, r r'), (d''_1, r'')\} + \partial_{1*}(d''_1, r'') \cdot \{(d_1, r), (d'_1, r')\} \\
 &= \{(d_1, r)(d'_1, r'), (d''_1, r'')\} + \partial_{1*}(d''_1, r'') \cdot \{(d_1, r), (d'_1, r')\}.
 \end{aligned}$$

PL4:

$$\begin{aligned}
 &\{(d_1, r), \partial_{2*}(d_2, r')\} + \{\partial_{2*}(d_2, r'), (d_1, r)\} \\
 &= \{(d_1, r), (\partial_2(d_2), r')\} + \{(\partial_2(d_2), r'), (d_1, r)\} \\
 &= (\{d_1, \partial_2(d_2)\}, r r') + (\{\partial_2(d_2), d_1\}, r' r) \\
 &= (\{d_1, \partial_2(d_2)\} + \{\partial_2(d_2), d_1\}, r r') \\
 &= (\partial_1(d_1) \cdot d_2, r r') \\
 &= (d_2, \phi(\partial_1(d_1)) r r') \\
 &= (\phi(\partial_1(d_1)) r) \cdot (d_2, r') \\
 &= \partial_{1*}(d_1, r) \cdot (d_2, r').
 \end{aligned}$$

PL5:

$$\begin{aligned}
 \{(d_1, r) \cdot r'', (d'_1, r')\} &= \{(d_1, r r''), (d'_1, r')\} \\
 &= (\{d_1, d'_1\}, r r' r') \\
 &= (\{d_1, d'_1\}, r r') \cdot r'' \\
 &= \{(d_1, r), (d'_1, r')\} \cdot r'' \\
 \{(d_1, r), (d'_1, r') \cdot r''\} &= \{(d_1, r), (d'_1, r' r'')\} \\
 &= (\{d_1, d'_1\}, r r' r'') \\
 &= (\{d_1, d'_1\}, r r') \cdot r'' \\
 &= \{(d_1, r), (d'_1, r')\} \cdot r''
 \end{aligned}$$

for all $(d_1, r), (d'_1, r'), (d''_1, r'') \in \phi_*(D_1)$, $(d_2, r') \in \phi_*(D_2)$ and $r'' \in R$.

The proof of proposition 11
PL3:

$$\begin{aligned}
 & \{d_1 + KD_1, (d'_1 + KD_1)(d''_1 + KD_1)\} \\
 = & \{d_1 + KD_1, (d'_1 d''_1 + KD_1)\} \\
 = & \{d_1, d'_1 d''_1\} + KD_2 \\
 = & (\{d_1 d'_1, d''_1\} + \partial_1(d''_1) \cdot \{d_1, d'_1\}) + KD_2 \\
 = & (\{d_1 d'_1, d''_1\} + KD_2) + (\partial_1(d''_1) \cdot \{d_1, d'_1\} + KD_2) \\
 = & (\{d_1 d'_1, d''_1\} + KD_2) + (\partial_1(d''_1) + K) \cdot \{d_1, d'_1\} + KD_2 \\
 = & (\{d_1 d'_1, d''_1\} + KD_2) + \partial_{1*}(d''_1 + KD_1) \cdot (\{d_1, d'_1\} + KD_2) \\
 = & \{d_1 d'_1 + KD_1, d''_1 + KD_1\} + \partial_{1*}(d''_1 + KD_1) \cdot \{d_1 + KD_1, d'_1 + KD_1\} \\
 = & \{(d_1 + KD_1)(d'_1 + KD_1), d''_1 + KD_1\} + \partial_{1*}(d''_1 + KD_1) \cdot \{d_1 + KD_1, d'_1 + KD_1\}.
 \end{aligned}$$

PL4:

$$\begin{aligned}
 & \{d_1 + KD_1, \partial_{2*}(d_2 + KD_2)\} + \{\partial_{2*}(d_2 + KD_2), d_1 + KD_1\} \\
 = & \{d_1 + KD_1, \partial_2(d_2) + KD_1\} + \{\partial_2(d_2) + KD_1, d_1 + KD_1\} \\
 = & (\{d_1, \partial_2(d_2)\} + KD_2) + (\{\partial_2(d_2), d_1\} + KD_2) \\
 = & (\{d_1, \partial_2(d_2)\} + \{\partial_2(d_2), d_1\}) + KD_2 \\
 = & (\partial_1(d_1) \cdot d_2) + KD_2 \\
 = & (\partial_1(d_1) + K) \cdot (d_2 + KD_2) \\
 = & \partial_{1*}(d_1 + KD_1) \cdot (d_2 + KD_2).
 \end{aligned}$$

PL5:

$$\begin{aligned}
 \{d_1 + KD_1, d'_1 + KD_1\} \cdot r &= \{d_1 + KD_1, d'_1 + KD_1\} \cdot (s + K) \\
 &= (\{d_1, d'_1\} + KD_2) \cdot (s + K) \\
 &= (\{d_1, d'_1\} \cdot s) + KD_2 \\
 &= (\{d_1 \cdot s, d'_1\}) + KD_2 \\
 &= \{(d_1 \cdot s) + KD_1, d'_1 + KD_1\} \\
 &= \{(d_1 + KD_1) \cdot (s + K), d'_1 + KD_1\} \\
 &= \{(d_1 + KD_1) \cdot r, d'_1 + KD_1\}. \\
 \\
 \{d_1 + KD_1, d'_1 + KD_1\} \cdot r &= \{d_1 + KD_1, d'_1 + KD_1\} \cdot (s + K) \\
 &= (\{d_1, d'_1\} + KD_2) \cdot (s + K) \\
 &= (\{d_1, d'_1\} \cdot s) + KD_2 \\
 &= (\{d_1, d'_1 \cdot s\}) + KD_2 \\
 &= \{d_1 + KD_1, (d'_1 \cdot s) + KD_1\} \\
 &= \{d_1 + KD_1, (d'_1 + KD_1) \cdot (s + K)\} \\
 &= \{d_1 + KD_1, (d'_1 + KD_1) \cdot r\}.
 \end{aligned}$$

for all $d_1 + KD_1, d'_1 + KD_1, d''_1 + KD_1 \in D_1/KD_1, d_2 + KD_2 \in D_2/KD_2, r \in R$ and $s + K \in S/K$.

Let us check that $(\phi'', \phi', \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{D_2/KD_2, D_1/KD_1, R, \partial_{2*}, \partial_{1*}\}$ where $\phi''(d_2) = d_2 + KD_2$ and $\phi''(d_1) = d_1 + KD_1$ is a morphism of 2-crossed modules.

$$\begin{array}{ccccc}
 D_2 & \xrightarrow{\partial_2} & D_1 & \xrightarrow{\partial_1} & S \\
 \phi'' \downarrow & & \phi' \downarrow & & \phi \downarrow \\
 D_2/KD_2 & \xrightarrow{\partial_{2*}} & D_1/KD_1 & \xrightarrow{\partial_{1*}} & R
 \end{array}$$

$$\begin{aligned}
 \phi''(s \cdot d_2) &= (s \cdot d_2) + KD_2 & \phi'(s \cdot d_1) &= (s \cdot d_1) + KD_1 \\
 &= (s + K) \cdot (d_2 + KD_2) & \text{and} & & = (s + K) \cdot (d_1 + KD_1) \\
 &= \phi(s) \cdot \phi''(d_2) & & & = \phi(s) \cdot \phi'(d_1)
 \end{aligned}$$

$$\begin{aligned}
 \partial_{2*}(\phi''(d_2)) &= \partial_{2*}(d_2 + KD_2) \\
 &= \partial_2(d_2) + KD_1 \\
 &= \phi'(\partial_2(d_2))
 \end{aligned}$$

similarly $\partial_{1*}\phi' = \phi\partial_1$ for all $d_1 \in D_1, d_2 \in D_2$ and $s \in S$.

$$\begin{aligned}
 (\{-, -\}(\phi' \times \phi'))(d_1, d'_1) &= \{-, -\}(d_1 + KD_1, d'_1 + KD_1) \\
 &= \{d_1, d'_1\} + KD_2 \\
 &= \phi''(\{d_1, d'_1\}) \\
 &= (\phi''\{-, -\})(d_1, d'_1).
 \end{aligned}$$

for all $d_1, d'_1 \in D_1$.

Acknowledgements

This work was partially supported by TÜBİTAK (The Scientific and Technological Research Council Of Turkey) for the second author.

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